Applications of the Intermediate Value Theorem
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Consider two firms A and B.

- When A chooses to produce x units of commodity, B's best response is producing $y = f(x)$ units which gives itself maximum profit. When B chooses to produce y units of commodity, A's best response is producing $x = g(y)$ units which gives itself maximum profit. Usually the best response functions, $f(x)$ and $g(y)$, are continuous.
- We are looking for a *Nash equilibrium*, $(\bar{x}, \bar{y})$, such that $\bar{y} = f(\bar{x})$ and $\bar{x} = g(\bar{y})$ which means that both A and B are best responding to the other's strategy.
- The Intermediate Value Theorem can help us to show that a Nash equilibrium exists!!

Exercise:
Suppose that there are two oligopolists, A and B, producing x and y units of a commodity, respectively. The price of the commodity is

$$P(x, y) = 80 - \frac{1}{4}(x + y).$$

The cost function for A is

$$C_1(x) = 4x + 100.$$

The cost function for B is

$$C_2(y) = 3y + 125.$$

A has revenue $P(x, y) \cdot x$ and B has revenue $P(x, y) \cdot y$. The profit is revenue minus cost.

1. Write down the profit for A and B.

The profit of A is $x \cdot P(x,y) - C_1(x) = 80x - \frac{x}{4}(x+y) - 4x - 100$
$= -\frac{1}{4}x^2 + 76x - \frac{1}{4}xy - 100$

The profit of B is $y \cdot P(x,y) - C_2(y) = 80y - \frac{y}{4}(x+y) - 3y - 125$
$= -\frac{1}{4}y^2 + 77y - \frac{1}{4}xy - 125$

2. To find the best response function of A, view the profit of A as a polynomial of x with degree 2 and y as a constant. Complete the square and find x_{max} to maximize profit. x_{max} which depends on y is the best response function of A, which we denote as $g(y)$.

$-\frac{1}{4}x^2 + 76x - \frac{1}{4}xy - 100 = -\frac{1}{4}(x^2 + xy - 304x) - 100$
$= -\frac{1}{4}\left[x + (\frac{y}{2} - 152)\right]^2 - 100 + \frac{1}{4}(\frac{y}{2} - 152)^2$

The maximum profit occurs at $x_{max} = 152 - \frac{y}{2}$.
Hence $g(y) = 152 - \frac{y}{2}$.

3. Similarly, find the best response function of B, $y_{max} = f(x)$.

$$-\frac{1}{4}y^2+77y-\frac{1}{4}xy-125 = -\frac{1}{4}\left(y^2+xy-308y\right)-125$$

$$= -\frac{1}{4}\left[y+\left(\frac{x}{2}-154\right)\right]^2 - 125+\frac{1}{4}\left(\frac{x}{2}-154\right)^2$$

The maximum profit occurs at $y_{max} = 154-\frac{x}{2}$.

Hence $f(x) = 154-\frac{x}{2}$.

4. Let A and B each be able to produce at most 160 units. Is there a Nash equilibrium $(\bar{x}, \bar{y})$ where $0 \leq \bar{x}, \bar{y} \leq 160$?

Solve the system of equations

$$\begin{cases} \bar{x} = g(\bar{y}) = 152-\frac{1}{2}\bar{y} \\ \bar{y} = f(\bar{x}) = 154-\frac{1}{2}\bar{x} \end{cases} \Rightarrow \begin{cases} \bar{x}+\frac{\bar{y}}{2} = 152 \\ \bar{y}+\frac{\bar{x}}{2} = 154 \end{cases} \Rightarrow \begin{matrix} \bar{x} = 100 \\ \bar{y} = 104 \end{matrix}.$$

$(\bar{x}, \bar{y}) = (100, 104)$ is a Nash equilibrium.

5. In general, suppose that best response functions $f(x)$ and $g(y)$ are continuous and there is some interval $[\alpha, \beta]$ such that the ranges of f and g on $[\alpha, \beta]$ are still contained in $[\alpha, \beta]$. Show that there exists a Nash equilibrium $(\bar{x}, \bar{y})$ such that $\alpha \leq \bar{x}, \bar{y} \leq \beta$.
(Hint: Show that the function $g \circ f$ has a *fixed point* $x_0 \in [\alpha, \beta]$, i.e. $g(f(x_0)) = x_0$, by the intermediate value theorem. Hence, $(\bar{x}, \bar{y}) = (x_0, f(x_0))$ is a Nash equilibrium.)

Consider the function $h(x) = g(f(x)) - x$.

$\because$ $f(x)$ and g are continuous function with domain $[\alpha, \beta]$ and range $\subset [\alpha, \beta]$.

$\therefore$ $h(x)$ is continuous on $[\alpha, \beta]$.

Moreover, since the range of g is contained in $[\alpha, \beta]$, we have $\alpha \leq g(f(\alpha)) \leq \beta$ and $h(\alpha) = g(f(\alpha)) - \alpha \geq 0$.

Similarly, $\alpha \leq g(f(\beta)) \leq \beta$ and $h(\beta) = g(f(\beta)) - \beta \leq 0$.

If $h(\alpha) = 0$ or $h(\beta) = 0$, then α or β is a fixed point of $g(f(x))$. If $h(\alpha) > 0$ and $h(\beta) < 0$, then by the intermediate value theorem, there is some $x_0 \in [\alpha, \beta]$ such that $h(x_0) = 0$ i.e. x_0 is a fixed point of $g(f(x))$.

Thus $g(f(x))$ must have a fixed point on $[\alpha, \beta]$, say x_0. Then let $y_0 = f(x_0)$. We have $g(y_0) = g(f(x_0)) = x_0$ and $y_0 = f(x_0)$ which means that (x_0, y_0) is a Nash equilibrium.